

Module sectional category of products

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Abstract

Extending a result of Félix-Halperin-Lemaire on Lusternik-Schnirelmann category of products, we prove additivity of a rational approximation for Schwarz’s sectional category with respect to products of fibrations.

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Introduction

The sectional category[11] (or Schwarz genus) of a fibration $p: E \rightarrow X$ is the smallest integer n such that X admits a cover by open sets on each of which a local section for p exists. This homotopy invariant is a generalization of the well known Lusternik-Schnirelmann (LS) category[9] of a path-connected space X , $\text{cat}(X)$, as it is the sectional category of the path fibration $PX \rightarrow X$, $\alpha \mapsto \alpha(1)$, where PX is the space of paths starting at the base point.

One of the most important results of [4] says that, if X and Y are simply connected rational spaces of finite type, then $\text{cat}(X \times Y) = \text{cat}(X) + \text{cat}(Y)$. This was done through Hess’ theorem [8] by proving the analogous result for

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a lower bound of LS category called module LS category.

Throughout this paper we will consider all spaces to be simply connected CW-complexes of finite type. We will also denote f_0 the rationalisation of a map f . As for LS category, there exists a lower bound of sectional category, called module sectional category [6], for which we have $\text{mcat}(X) = \text{msecat}(PX \rightarrow X)$. In this paper we prove

Theorem 1. *Let f and g be two fibrations. If f_0 admits a homotopy retraction, then*

$$\text{msecat}(f \times g) = \text{msecat}(f) + \text{msecat}(g).$$

Another important particular case of sectional category is Farber's (higher) topological complexity [3, 10] of a space X , $\text{TC}_n(X) = \text{secat}(\pi_n)$, where the fibration $\pi_n: X^{[1,n]} \rightarrow X^n$ is such that $\pi_n(\alpha) = (\alpha(1), \alpha(2), \dots, \alpha(n))$.

As a direct application of Theorem 1, the module invariant associated to (higher) topological complexity,

$$\text{mTC}_n(X) := \text{msecat}(\pi_n),$$

is additive:

Corollary 2. *Let X and Y be two spaces. Then*

$$\text{mTC}_n(X \times Y) = \text{mTC}_n(X) + \text{mTC}_n(Y).$$

The results given are an improvement of [2].

1 Preliminaries

This section contains a brief summary of the tools that will be used, see [5] for further details. Let (A, d) be a commutative differential graded algebra over \mathbb{Q} (**cdga**). An (A, d) -module is a chain complex (M, d) together with a degree 0 action of A verifying that $d(ax) = d(a)x + (-1)^{\deg(a)}ad(x)$. The module $M^\# = \text{hom}(M, \mathbb{Q})$ admits an (A, d) -module structure with action $(a\varphi)(x) = (-1)^{\deg(a)\deg(\varphi)}\varphi(ax)$ and differential $d(\varphi) = (-1)^{\deg(\varphi)}\varphi \circ d$. If N is an (A, d) -modules, then the module $M \otimes_A N$ admits an (A, d) -module structure with action $a(m \otimes n) = (am) \otimes n$ and differential $d(m \otimes n) =$

$d(m) \otimes n + (-1)^{\deg(m)} m \otimes d(n)$. An (A, d) -module P is said to be semifree if there exists an increasing filtration P_* by (A, d) -submodules such that P_k/P_{k-1} is a free (A, d) module on a basis of cocycles. Every (A, d) -module M admits a semifree resolution, that is a quasi-isomorphism of (A, d) modules, $P \xrightarrow{\simeq} M$, where P is (A, d) -semifree. If P is (A, d) -semifree and if η is a quasi-isomorphism of (A, d) -modules then $\eta \otimes_A \text{Id}_P$ and $\text{Id}_P \otimes_A \eta$ are also quasi-isomorphisms. A morphism of (A, d) -modules $\varphi: (M, d) \rightarrow (N, d)$ is said to have a homotopy retraction if there exists a commutative diagram of (A, d) -modules,

$$\begin{array}{ccc} (M, d) & \xrightarrow{\text{Id}} & (M, d) \\ \varphi \downarrow & \searrow & \uparrow \\ (N, d) & \xleftarrow{\simeq} & (P, d). \end{array}$$

We will use the following lemma which is an expression of one of the central ideas of [4].

Lemma 3. *Let $\varphi: (A, d) \rightarrow (B, d)$ be a surjective **cdga** morphism with kernel K and A of finite type. The morphism φ admits a homotopy retraction of (A, d) -modules if and only if for any (A, d) semi-free resolution $\eta: P \xrightarrow{\simeq} A^\#$, the projection*

$$\varrho: P \longrightarrow \frac{P}{K \cdot P}$$

is injective in homology.

Proof. Suppose that φ admits a homotopy retraction of (A, d) -module. This means that there exists a commutative diagram of (A, d) module of the form

$$\begin{array}{ccc} A & \xrightarrow{\text{Id}_A} & A \\ \varphi \downarrow & \searrow i & \uparrow r \\ B & \xleftarrow{\simeq} & M \end{array}$$

where we can suppose that M is a (A, d) semi-free resolution. Let now $P \xrightarrow{\simeq} A^\#$ be a (A, d) semi-free resolution and apply $-\otimes_A P$ to the diagram above. We get

$$\begin{array}{ccc} P & \xrightarrow{\text{Id}_P} & P \\ \downarrow & \searrow & \uparrow \\ B \otimes_A P & \xleftarrow{\simeq} & M \otimes_A P. \end{array}$$

Since $B = \frac{A}{K}$ we have $B \otimes_A P = \frac{P}{K \cdot P}$ and the left hand morphism is the projection $\varrho: P \rightarrow \frac{P}{K \cdot P}$. The diagram shows that ϱ admits a homotopy retraction of (A, d) -module and therefore that it is injective in homology.

Conversely, suppose that ϱ is homology injective. Since A is of finite type, $\eta^\# = \text{hom}(\eta, \mathbb{Q}): A \rightarrow \text{hom}(P, \mathbb{Q})$ is also a semi-free resolution. Since ϱ is homology injective,

$$\varrho^\# = \text{hom}(\varrho, \mathbb{Q}): \text{hom}\left(\frac{P}{K \cdot P}, \mathbb{Q}\right) \rightarrow \text{hom}(P, \mathbb{Q})$$

is homology surjective. There exists then a cycle $f \in \text{hom}\left(\frac{P}{K \cdot P}, \mathbb{Q}\right)$ such that $[f \circ \varrho] = [z]$, where $z = \eta^\#(1)$. Now define the (A, d) -module morphism $\alpha: A \rightarrow \text{hom}\left(\frac{P}{K \cdot P}, \mathbb{Q}\right)$ as $\alpha(1) = f$. Then $\varrho^\# \circ \alpha$ is homotopic to $\eta^\#$ and thus a quasi-isomorphism. To finish the proof, we observe that $K \cdot \text{hom}\left(\frac{P}{K \cdot P}, \mathbb{Q}\right) = \{0\}$ and thus we have a diagram

$$\begin{array}{ccccc} A & \xrightarrow{\text{Id}_A} & A & & \\ \varphi \downarrow & & \simeq \downarrow \varrho^\# \circ \alpha & & \\ B & \xrightarrow{\bar{\alpha}} & \text{hom}\left(\frac{P}{K \cdot P}, \mathbb{Q}\right) & \xrightarrow{\varrho^\#} & \text{hom}(P, \mathbb{Q}), \end{array}$$

which yields a homotopy retraction for φ as (A, d) -modules. \square

Let us denote by $p_n: J_X^n(E) \rightarrow X$ the join of $n + 1$ copies of a fibration $p: E \rightarrow X$. As it is well-known [11], $\text{secat}(p) \leq n$ if and only if p_n admits a homotopy section. By definition, $\text{msecat}(p)$ is the smallest n such that $A_{PL}(p_n)$ admits a homotopy retraction of $A_{PL}(X)$ -modules, where A_{PL} denotes Sullivan's functor of piecewise linear forms[12].

Recall the following general characterization of $\text{msecat}(f)$ from [6]. Let $(A, d) \rightarrow (A \otimes (\mathbb{Q} \oplus X), d)$ be a semi-free extension of (A, d) -module which is a model for f . For $x \in X$, write $dx = d_0x + d_+x \in A \oplus A \otimes X$. Then $\text{msecat}(f)$ is the least m such that the following (A, d) semi-free extension admits a retraction of (A, d) -module:

$$j_m: (A, d) \rightarrow J_m = (A \otimes (\mathbb{Q} \oplus s^{-m} X^{\otimes m+1}), d).$$

Here $d = d_0 + d_+$ (in $A \oplus A \otimes s^{-m}X^{\otimes m+1}$) is given by

$$\begin{aligned} d(s^{-m}x_0 \otimes \cdots \otimes x_m) &= (-1)^{\sum_{k=1}^m (k|x_{m-k}|+k-1)} d_0x_0 \cdots d_0x_m \\ &+ \sum_{i=0}^m \sum_{j_i} (-1)^{(|a_{ij_i}|+1)(|x_0|+\cdots+|x_{i-1}|+m)} a_{ij_i} \otimes s^{-m}x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_m, \end{aligned}$$

for $x_0, \dots, x_m \in X$ and $d_+x_i = \sum_{j_i} a_{ij_i} \otimes x_{ij_i}$ with $a_{ij_i} \in A$ and $x_{ij_i} \in X$.

Using the following notation (suggested by the standard rules of signs)

$$s^{-m}x_0 \otimes \cdots \otimes d_+x_i \otimes \cdots \otimes x_m := \sum_{j_i} \sigma_{ij_i} a_{ij_i} \otimes s^{-m}x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_m$$

we can write $d_+(s^{-m}x_0 \otimes \cdots \otimes x_m)$ as

$$d_+(s^{-m}x_0 \otimes \cdots \otimes x_m) = (-1)^m \sum_{i=0}^m \sum_{j_i} \tau_i s^{-m}x_0 \otimes \cdots \otimes d_+x_i \otimes \cdots \otimes x_m,$$

where $\sigma_{ij_i} := (-1)^{|a_{ij_i}|(|x_0|+\cdots+|x_{i-1}|+m)}$ and $\tau_i := (-1)^{(|x_0|+\cdots+|x_{i-1}|)}$.

Now let f be a fibration such that f_0 admits a homotopy retraction. Then by [1], there exists a surjective model for f , $\varphi: A \rightarrow \frac{A}{K}$ (called s-model) such that $\text{msecat}(f)$ is the smallest m for which the projection $\rho_m: A \rightarrow \frac{A}{K^{m+1}}$ admits a homotopy retraction of (A, d) -modules. We have

Proposition 4. *Let f be a fibration such that f_0 admits a homotopy retraction, $\varphi: A \rightarrow \frac{A}{K}$ and s-model for f and $(A, d) \rightarrow (A \otimes (\mathbb{Q} \oplus X), d)$ a semifree model for f , as in previous paragraphs. Let also $\eta: P \xrightarrow{\simeq} A^\#$ be an (A, d) semi-free resolution. Then the following are equivalent*

- (i) $\text{msecat}(f) \leq m$,
- (ii) the morphism $\text{Id}_P \otimes_A j_m: P \rightarrow P \otimes (\mathbb{Q} \oplus s^{-m}X^{\otimes m+1})$ is injective in homology,
- (iii) the projection $P \rightarrow \frac{P}{K^{m+1}.P}$ is injective in homology.

Proof. By [1], there is a diagram

$$\begin{array}{ccc} & A & \\ j_m \swarrow & \downarrow h_A & \searrow \\ J_m & \xrightarrow{\simeq} C & \xleftarrow{\quad} \frac{A}{K^{m+1}} \end{array}$$

where the left hand triangle is commutative up to a homotopy of (A, d) -modules and the right hand triangle is strictly commutative. Applying to previous diagram $\text{Id}_P \otimes_A -$, we get the following diagram of (A, d) module:

$$\begin{array}{ccc} & P & \\ \text{Id}_P \otimes_A j_m \swarrow & \downarrow & \searrow \\ P \otimes (\mathbb{Q} \oplus s^{-m} X^{\otimes m+1}) & \xrightarrow{\simeq} P \otimes_A C & \xleftarrow{\quad} \frac{P}{K^{m+1}, P} \end{array}$$

where the left hand triangle is commutative up to a homotopy of (A, d) -module and the right hand triangle is strictly commutative. The result then follows from Lemma 3. \square

2 The main result

Observe that Proposition 4 together with the strategy of [4] can be used to easily prove Theorem 1 provided that both fibrations admit a homotopy retractions. In line with our statement, we here present a proof of the additivity of module sectional category when only one of the fibrations admits homotopy retraction.

We first notice that one of the inequalities of Theorem 1 follows in general:

Proposition 5. *Let $p: E \rightarrow X$ and $p': E' \rightarrow X'$ be two fibrations. We have*

$$\text{msecat}(p \times p') \leq \text{msecat}(p) + \text{msecat}(p').$$

Proof. In [7, Pg. 26], a commutative diagram of the following form is constructed:

$$\begin{array}{ccc} J_X^n(E) \times J_{X'}^m(E') & \xrightarrow{\psi_{n,m}^{E,E'}} & J_{X \times X'}^{m+n}(E \times E') \\ & \searrow p_n \times p'_m & \swarrow (p \times p')_{n+m} \\ & X \times X' & \end{array}$$

By applying A_{PL} to this diagram, we can establish that, if $\text{msecat}(p) \leq n$ and $\text{msecat}(p') \leq m$ then $\text{msecat}(p \times p') \leq n + m$. \square

Keeping the notation of Proposition 4, the differential in $(A \otimes (\mathbb{Q} \oplus X), d)$ can be taken such that $d_0(x) \in K$. This implies that, in $P \otimes (\mathbb{Q} \oplus s^{-m} X^{\otimes m+1})$, $d_0(s^{-m} X^{\otimes m+1}) \subset K^{m+1} \cdot P$. With this in mind we proceed to the

Proof of Theorem 1. Take an s-model for f , φ and an (A, d) semi-free extension of φ , $A \otimes (\mathbb{Q} \oplus X)$, as in the previous section. Let also $(B, d) \rightarrow (B \otimes (\mathbb{Q} \oplus Y), d)$ a semi-free model of g (where (B, d) is a **cdga** model of the base of g). Then $f \times g$ is modeled by the tensor product of the two semi-free extensions which gives a semi-free extension of $(A \otimes B, d)$ -modules that we write as follows

$$A \otimes B \rightarrow A \otimes B \otimes (\mathbb{Q} \oplus Z) \quad \text{where } Z = X \oplus Y \oplus X \otimes Y.$$

In order to prove the statement, we suppose $\text{msecat}(f) = m$ and $\text{msecat}(f \times g) \leq m + p$ and we establish that $\text{msecat}(g) \leq p$.

Let $P \xrightarrow{\simeq} A^\#$ be an (A, d) semi-free resolution. Since $\text{msecat}(f) = m$ we know from Proposition 4 that there exists $\Omega \in H(K^m \cdot P)$ which is not trivial in $H(P)$. Then there exist a cocyle $\omega \in K^m \cdot P$ representing Ω in $H(P)$ and $\theta \in P \otimes s^{-(m-1)} X^{\otimes m}$ such that $d\theta = \omega$. As a chain complex, we can write $P = \omega \cdot \mathbb{Q} \oplus S$ where $d(S) \subset S$, and we define the following linear map of degree $-|\omega|$:

$$I_\omega: P \rightarrow \mathbb{Q}, \quad I_\omega(\omega) = 1, \quad I_\omega(S) = 0.$$

This map commutes with differentials. Now write the element $\theta \in P \otimes s^{-(m-1)} X^{\otimes m}$ as

$$\theta = \sum_i m_i \otimes s^{-(m-1)} x_i$$

with $m_i \in P$ and $x_i \in X^{\otimes m}$. Since $d\theta = \omega$ we have $d_+ \theta = 0$ and $d_0 \theta = \omega$.

Let $\psi: B \otimes (\mathbb{Q} \oplus s^{-p} Y^{\otimes p+1}) \rightarrow P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-p} Z^{\otimes m+p+1})$ be the B -linear map of degree $|\omega|$ given by $\psi(1) = \omega \otimes 1$ and, for $y \in Y^{\otimes p+1}$,

$$\psi(s^{-p} y) = -(-1)^{p|\omega|} \sum_i (-1)^{(p+1)|m_i|} m_i \otimes 1 \otimes s^{-m-p} x_i \otimes y$$

and extended to $B \otimes (\mathbb{Q} \oplus s^{-p} Y^{\otimes p+1})$ by the rule $\psi(b \cdot x) = (-1)^{|b||\omega|} b \cdot \psi(x)$. Notice that the structure of (B, d) -module on $P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-p} Z^{\otimes m+p+1})$ is

given by $b \cdot (m \otimes b' \otimes z) = (-1)^{|m||b|} m \otimes b b' \otimes z$. In particular $\psi(b) = \omega \otimes b$. Let us now see that ψ commutes with differentials, that is $\psi \circ d = (-1)^{|\omega|} d \circ \psi$. Since ψ is B -linear and since ω is a cocycle we only have to see that

$$d\psi(s^{-p}y) = (-1)^{|\omega|} \psi(ds^{-p}y),$$

for each $y \in Y^{\otimes p+1}$. Writing the differential of $P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-p} Z^{\otimes m+p+1})$ as

$$d = d_0 + d_+ \in P \otimes B \oplus P \otimes B \otimes s^{-m-p} Z^{\otimes m+p+1}$$

we can check that

- $d_0\psi(s^{-p}y) = (-1)^{|\omega|} \psi(d_0 s^{-p}y)$ using the fact that $d_0\theta = \omega$, and
- $d_+\psi(s^{-p}y) = (-1)^{|\omega|} \psi(d_+ s^{-p}y)$ using the fact that $d_+\theta = 0$.

From $\text{msecat}(f \times g) \leq m + p$ we know that the morphism

$$j_{m+p}^{A \otimes B}: A \otimes B \rightarrow A \otimes B \otimes (\mathbb{Q} \oplus s^{-m-p} Z^{\otimes m+p+1})$$

admits a retraction r of $(A \otimes B, d)$ -modules. Finally the composite

$$\begin{array}{ccc} B \otimes (\mathbb{Q} \oplus s^{-p} Y^{\otimes p+1}) & \xrightarrow{\psi} & P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-p} Z^{\otimes m+p+1}) \\ & & \downarrow P \otimes_A r \\ & & P \otimes B \xrightarrow{I_\omega \otimes \text{Id}} B \end{array}$$

gives a morphism (of degree 0) of (B, d) -module which is a retraction for the inclusion $B \rightarrow B \otimes (\mathbb{Q} \oplus s^{-p} Y^{\otimes p+1})$. This proves that $\text{msecat}(g) \leq p$. \square

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